

# Einstein-Yang-Mills black hole solution in higher dimensions by the Wu-Yang Ansatz

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## Abstract

By employing the higher ( $N \geq 5$ ) dimensional version of the Wu-Yang Ansatz we obtain black hole solutions in the spherically symmetric Einstein-Yang-Mills (EYM) theory. Although these solutions were found recently by other means, our method provides an alternative way in which one identifies the contribution from the Yang-Mills (YM) charge. Our method has the advantage to be carried out analytically as well. We discuss some interesting features of the black hole solutions obtained.

## 1 Introduction

Black holes in Einstein and Einstein-Maxwell (EM) theory have a long history, starting with Schwarzschild as early as 1916 which is a well-known subject by now. Higher dimensional version of black holes has also attracted attention during the last three decades [1,2]. Extension of this fascinating subject to the Einstein-Yang-Mills (EYM) theory is relatively rather new[3-5]. This originates from the intricate structure of the YM system compared with the Maxwell's electromagnetism. Finding exact solutions to the YM system in a flat background is a challenging problem by itself, not to mention its coupled form with gravity. In spite of all its inherent complications when specified to spherical symmetry and specific gauge group such as  $SO(N-1)$  analytical solutions can be obtained. From this token black hole solutions have been obtained newly[4-7].

Our aim in this Letter is to show that black hole solutions can be obtained by an alternative method, namely by making use of the Wu-Yang Ansatz [8-9] in higher ( $N > 4$ ) dimensions. Our method has the advantage of introducing the YM charge ab initio and obtain, after integrating the EYM equations, the

charge dependent term in the metric function. Except for the  $N = 5$  case, the EYM equations dictate a Reissner-Nordstrom type black hole solution. The case  $N = 5$ , however, leads to a black hole which incorporates a logarithmic term unprecedented in the  $N > 5$  cases. For this reason we shall treat the case  $N = 5$  in some detail while  $N > 5$  cases will be considered separately.

## 2 Field equations in Einstein-Yang-Mills theory

The action which describes Einstein-Yang-Mills gravity without a cosmological constant in  $N$  dimensions reads

$$I_G = \frac{1}{2} \int_{\mathcal{M}} dx^N \sqrt{-g} \left[ R - \sum_{a=1}^{(N-1)(N-2)/2} F_{\mu\nu}^{(a)} F^{(a)\mu\nu} \right] \quad (1)$$

where  $R$  is the Ricci Scalar and the YM fields  $F_{\mu\nu}^{(a)}$  are

$$F_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + \frac{1}{2\sigma} C_{(b)(c)}^{(a)} A_\mu^{(b)} A_\nu^{(c)} \quad (2)$$

where  $C_{(b)(c)}^{(a)}$  are the structure constants of the  $\frac{(N-1)(N-2)}{2}$ -parameter Lie group  $G$ ,  $\sigma$  is a coupling constant, and  $A_\mu^{(a)}$  are the gauge potentials. We note that the internal indices  $\{a, b, c, \dots\}$  do not differ whether in covariant or contravariant form. Variation of the action with respect to the space-time metric  $g_{\mu\nu}$  yields the EYM equations

$$G_{\mu\nu} = T_{\mu\nu}, \quad (3)$$

where the stress-energy tensor is

$$T_{\mu\nu} = \sum_{a=1}^{(N-1)(N-2)/2} \left[ 2F_\mu^{(a)\lambda} F_{\nu\lambda}^{(a)} - \frac{1}{2} F_{\lambda\sigma}^{(a)} F^{(a)\lambda\sigma} g_{\mu\nu} \right]. \quad (4)$$

Variation of the action with respect to the gauge potentials  $A_\mu^{(a)}$  yields the Yang-Mills equations

$$F_{;\mu}^{(a)\mu\nu} + \frac{1}{\sigma} C_{(b)(c)}^{(a)} A_\mu^{(b)} F^{(c)\mu\nu} = 0 \quad (5)$$

while the integrability conditions are

$$*F_{;\mu}^{(a)\mu\nu} + \frac{1}{\sigma} C_{(b)(c)}^{(a)} A_\mu^{(b)} *F^{(c)\mu\nu} = 0 \quad (6)$$

in which  $*$  means duality [10].

## 3 5D EYM black holes

Unlike the more general form of the 5-dimensional spherically symmetric line element, given by Brihaye et al [4], we restrict ourselves to work with a symmetric

version of it [6], to gain the benefits of the Wu-Yang Ansatz in Yang-Mills equations (i.e., this symmetric Ansatz provides us to cancel the role of gravity in YM equations, which means that, the metric function  $f(r)$  will not appear in the YM equations) as

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_3^2 \quad (7)$$

where

$$d\Omega_3^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \sin^2 \theta \sin^2 \phi d\psi^2 \quad (8)$$

in which

$$0 \leq \theta \leq \pi, 0 \leq \phi, \psi \leq 2\pi.$$

We introduce the new 5-dimentional Wu-Yang Ansatz [11] as follows

$$\begin{aligned} A^{(a)} &= \frac{Q}{r^2} (x_i dx_j - x_j dx_i) \\ 2 &\leq i \leq 4, \\ 1 &\leq j \leq i-1 \\ 1 &\leq (a) \leq 6 \end{aligned} \quad (9)$$

or explicitly

$$\begin{aligned} A^{(1)} &= \frac{Q}{r^2} (x_2 dx_1 - x_1 dx_2) \\ A^{(2)} &= \frac{Q}{r^2} (x_3 dx_1 - x_1 dx_3) \\ A^{(3)} &= \frac{Q}{r^2} (x_3 dx_2 - x_2 dx_3) \\ A^{(4)} &= \frac{Q}{r^2} (x_4 dx_1 - x_1 dx_4) \\ A^{(5)} &= \frac{Q}{r^2} (x_4 dx_2 - x_2 dx_4) \\ A^{(6)} &= \frac{Q}{r^2} (x_4 dx_3 - x_3 dx_4) \end{aligned} \quad (10)$$

where  $Q$  is the only non-zero gauge charge and

$$\begin{aligned} x_1 &= r \cos \psi \sin \phi \sin \theta \\ x_2 &= r \sin \psi \sin \phi \sin \theta \\ x_3 &= r \cos \phi \sin \theta \\ x_4 &= r \cos \theta. \end{aligned} \quad (11)$$

By setting  $\sigma = Q$  in Eq. (2), the YM field 2-forms are given by

$$F^{(a)} = dA^{(a)} + \frac{1}{2Q} C_{(b)(c)}^{(a)} A^{(b)} \wedge A^{(c)}. \quad (12)$$

We note that our notation follows the standard exterior differential forms, namely  $d$  stands for the exterior derivative, while  $\wedge$  stands for the wedge product. The hodge star  $*$  in the sequel will be used to represent duality [10].

The YM integrability conditions

$$dF^{(a)} + \frac{1}{Q} C_{(b)(c)}^{(a)} A^{(b)} \wedge F^{(c)} = 0 \quad (13)$$

are easily satisfied by using (10). The YM equations

$$d * F^{(a)} + \frac{1}{Q} C_{(b)(c)}^{(a)} A^{(b)} \wedge * F^{(c)} = 0 \quad (14)$$

are all satisfied as well.

The energy-momentum tensor

$$T_{\mu\nu} = \sum_{a=1}^6 \left[ 2F_{\mu}^{(a)\lambda} F_{\nu\lambda}^{(a)} - \frac{1}{2} F_{\lambda\sigma}^{(a)} F^{(a)\lambda\sigma} g_{\mu\nu} \right] \quad (15)$$

where  $\sum_{a=1}^6 \left[ F_{\lambda\sigma}^{(a)} F^{(a)\lambda\sigma} \right] = 6Q^2/r^4$ , has the non-zero components

$$\begin{aligned} T_{tt} &= \frac{3Q^2 f(r)}{r^4} \\ T_{rr} &= -\frac{3Q^2}{r^4 f(r)} \\ T_{\theta\theta} &= \frac{Q^2}{r^2} \\ T_{\phi\phi} &= \frac{Q^2 \sin^2 \theta}{r^2} \\ T_{\psi\psi} &= \frac{Q^2 \sin^2 \theta \sin^2 \phi}{r^2} \end{aligned} \quad (16)$$

The EYM equations  $G_{\mu\nu} = T_{\mu\nu}$ , reduce to the simple set of equations

$$\begin{aligned} r f'(r) + 2(f(r) - 1) &= \frac{-2Q^2}{r^2} \\ r^2 f''(r) + 4r f'(r) + 2(f(r) - 1) &= \frac{2Q^2}{r^2} \end{aligned} \quad (17)$$

in which a prime denotes derivative with respect to  $r$ .

This set admits the solution

$$f(r) = 1 - \frac{m}{r^2} - \frac{2Q^2}{r^2} \ln(r) \quad (18)$$

in which  $m$  is the usual integration constant to be identified as mass (this solution was obtained by Brihaye et al in different manner in reference [4]). The radii of Cauchy horizon  $r_-$  and the event horizon  $r_+$  are determined from the roots of  $f(r) = 0$ . For  $m$  and  $Q$  chosen outside of the unmarked region  $\mathcal{R}$  in the Fig. (1), one may get

$$r_- = \exp \left[ -\frac{1}{2Q^2} \left( m + Q^2 \text{LambertW} \left( 0, -\frac{e^{-\frac{m}{Q^2}}}{Q^2} \right) \right) \right] \quad (19)$$

$$r_+ = \exp \left[ -\frac{1}{2Q^2} \left( m + Q^2 \text{LambertW} \left( -1, -\frac{e^{-\frac{m}{Q^2}}}{Q^2} \right) \right) \right] \quad (20)$$

in which  $LambertW(k, x)$  is the Lambert function[12]. The asymptotic behavior of  $f(r)$ , which admits

$$\begin{aligned}\lim_{r \rightarrow 0^+} f(r) &= +\infty \\ \lim_{r \rightarrow \infty} f(r) &= 1\end{aligned}\tag{21}$$

also and having roots in between reconfirms the existence of  $r_+$  and  $r_-$  simultaneously. Fig. (1) clearly shows the possible black-hole region in  $m - Q$  plane. In the Fig. (2) we give the radii of the horizons in term of  $m$  while  $Q$  is set to be fixed, and in the Fig. (3) same thing is given but in term of  $Q$ , while  $m$  is fixed. These two figures clearly show that, with a large value of charge, the radii of the horizons are  $m$ -independent which is a consequence of logarithmic term in the metric function.

The surface gravity,  $\kappa$  defined by [13]

$$\kappa^2 = -\frac{1}{4}g^{tt}g^{ij}g_{tt,i}g_{tt,j}\tag{22}$$

has the value

$$\kappa = \left| \frac{1}{2}f'(r_+) \right| = \left| \frac{1}{r_+^3} (m - Q^2 + 2Q^2 \ln r_+) \right|.\tag{23}$$

The associated Hawking temperature is given by

$$T_H = \frac{\kappa}{2\pi} = \left| \frac{1}{2\pi r_+^3} (m - Q^2 + 2Q^2 \ln r_+) \right|\tag{24}$$

in the choice of units  $c = G = \hbar = k = 1$ .

## 4 The Wu-Yang Ansatz in $N > 5$ dimensions

The  $N$ -dimensional line element is chosen as

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{N-2}^2\tag{25}$$

in which the  $S^{N-2}$  line element will be expressed in the standard spherical form

$$d\Omega_{N-2}^2 = d\theta_1^2 + \sum_{i=2}^{N-3} \prod_{j=1}^{i-1} \sin^2 \theta_j d\theta_i^2\tag{26}$$

where

$$0 \leq \theta_1 \leq \pi, 0 \leq \theta_i \leq 2\pi.$$

We introduce the Wu-Yang Ansatz in N-D as

$$\begin{aligned}
A^{(a)} &= \frac{Q}{r^2} (x_i dx_j - x_j dx_i) \\
2 &\leq i \leq N-1, \\
1 &\leq j \leq i-1 \\
1 &\leq (a) \leq (N-1)(N-2)/2
\end{aligned} \tag{27}$$

where we imply (to have a systematic process) that the super indices  $a$  is chosen according to the values of  $i$  and  $j$  in order. For instance, we present some of them

$$\begin{aligned}
A^{(1)} &= \frac{Q}{r^2} (x_2 dx_1 - x_1 dx_2) \\
A^{(2)} &= \frac{Q}{r^2} (x_3 dx_1 - x_1 dx_3) \\
A^{(3)} &= \frac{Q}{r^2} (x_3 dx_2 - x_2 dx_3) \\
A^{(4)} &= \frac{Q}{r^2} (x_4 dx_1 - x_1 dx_4) \\
A^{(5)} &= \frac{Q}{r^2} (x_4 dx_2 - x_2 dx_4) \\
A^{(6)} &= \frac{Q}{r^2} (x_4 dx_3 - x_3 dx_4) \\
A^{(7)} &= \frac{Q}{r^2} (x_5 dx_1 - x_1 dx_5) \\
A^{(8)} &= \frac{Q}{r^2} (x_5 dx_2 - x_2 dx_5) \\
A^{(9)} &= \frac{Q}{r^2} (x_5 dx_3 - x_3 dx_5) \\
A^{(10)} &= \frac{Q}{r^2} (x_5 dx_4 - x_4 dx_5) \\
&\dots
\end{aligned} \tag{28}$$

in which  $r^2 = \sum_{i=1}^{N-1} x_i^2$ .

The YM field 2-forms are defined by the expression

$$F^{(a)} = dA^{(a)} + \frac{1}{2Q} C_{(b)(c)}^{(a)} A^{(b)} \wedge A^{(c)}. \tag{29}$$

In Sec.3 ( i.e.  $N = 5$ ), we had  $a = 1...6$ . For  $N = 6$  we have  $a = 1...10$ , and in general for  $N$  we must have  $(N-1)(N-2)/2$  gauge potentials. The integrability conditions

$$dF^{(a)} + \frac{1}{Q} C_{(b)(c)}^{(a)} A^{(b)} \wedge F^{(c)} = 0 \tag{30}$$

are easily satisfied by using (28). The YM equations

$$d * F^{(a)} + \frac{1}{Q} C_{(b)(c)}^{(a)} A^{(b)} \wedge * F^{(c)} = 0 \tag{31}$$

also are all satisfied. The energy-momentum tensor (4), becomes after

$$\sum_{a=1}^{(N-1)(N-2)/2} \left[ F_{\lambda\sigma}^{(a)} F^{(a)\lambda\sigma} \right] = \frac{(N-3)(N-2)Q^2}{r^4} \tag{32}$$

with the non-zero components

$$\begin{aligned}
T_{00} &= \frac{(N-3)(N-2)Q^2 f(r)}{2r^4} \\
T_{11} &= -\frac{(N-3)(N-2)Q^2}{2r^4 f(r)} \\
T_{22} &= -\frac{(N-3)(N-6)Q^2}{2r^2} \\
T_{AA} &= -\frac{(N-3)(N-6)Q^2}{2r^2} \prod_{i=1}^{A-2} \sin^2 \theta_i \\
2 &< A \leq N-1.
\end{aligned} \tag{33}$$

The EYM equations

$$G_{\mu\nu} = T_{\mu\nu}, \tag{34}$$

reduce to the set of general equations

$$\begin{aligned}
r^3 f'(r) + (N-3)r^2(f(r)-1) + (N-3)Q^2 &= 0 \\
r^4 f''(r) + 2(N-3)r^3 f'(r) + (N-3)(N-4)r^2(f(r)-1) + \\
(N-3)(N-6)Q^2 &= 0 \\
N &\geq 5
\end{aligned} \tag{35}$$

in which a prime denotes derivative with respect to  $r$ .

These equations admit the solution

$$\begin{aligned}
f(r) &= 1 - \frac{m}{r^{N-3}} - \frac{(N-3)Q^2}{(N-5)r^2} \\
N &> 5
\end{aligned} \tag{36}$$

in which  $m$  is the usual integration constant to be identified as mass. In the sequel as particular examples we consider the  $N=6$  and  $N>5$  cases in general.

#### 4.1 $N=6$ case

Equation (36) for  $N=6$  implies

$$f(r) = 1 - \frac{m}{r^3} - \frac{3Q^2}{r^2} \tag{37}$$

from which, since we consider  $m$  to be positive, one can only find the radius of the event horizon (i.e.,  $r_+$ ) by solving the following depressed cubic equation

$$r^3 - 3Q^2 r - m = 0 \tag{38}$$

in which the solution admits

$$r_+ = \frac{1}{2} \sqrt[3]{\Delta} + \frac{2Q^2}{\sqrt[3]{\Delta}} \tag{39}$$

where

$$\Delta = 4m + 4\sqrt{m^2 - 4Q^6}. \quad (40)$$

Following these results, we plot the Fig.s (4) and (5) to show how the radius of event horizon behaves in terms of the mass and the gauge charge.

Nevertheless the asymptotic behaviors of  $f(r)$  is given by

$$\lim_{r \rightarrow 0^+} f(r) = \lim_{r \rightarrow 0^+} \left(1 - \frac{m}{r^3}\right) = -\infty \quad (41)$$

and

$$\lim_{r \rightarrow \infty} f(r) = \lim_{r \rightarrow +\infty} \left(1 - \frac{3Q^2}{r^2}\right) = 1. \quad (42)$$

These imply that in 6-dimensional EYM-black hole, the Cauchy horizon  $r_-$  is not defined. We comment also that if one sets  $m = 0$  and  $Q \neq 0$ , similar to the 5-dimensional Schwarzschild black hole the metric function takes the form

$$f(r) = 1 - \frac{3Q^2}{r^2} \quad (43)$$

whose radius of the event horizon is given by

$$r_+ = \sqrt{3}Q. \quad (44)$$

## 4.2 Arbitrary $N > 5$ case

The metric function  $f(r)$  in any arbitrary dimensions  $N$  given by (36), has the following limits:

$$\begin{aligned} \lim_{r \rightarrow \infty} f(r) &= 1 \\ \lim_{r \rightarrow 0^+} f(r) &= -\infty. \end{aligned} \quad (45)$$

The radius of the event horizon may be determined by finding the root(s) of the metric function  $f(r)$ . It is not difficult to show that, with positive mass and  $N > 5$ ,  $f(r)$  has only one positive real root. This manifests itself as an essential difference between  $N = 5$  and  $N > 5$  dimensions in which, the black hole solution in five dimensions always has the Cauchy and event horizon, while for  $N > 5$ , the black hole solutions have only the event horizon.

In  $N$ -dimensional-EYM-black hole, when the mass of the black hole is zero (i.e.  $m = 0$ ) we have a gauge charged black hole whose radius of event horizon is given by

$$r_+ = Q\sqrt{\frac{N-3}{N-5}} \quad (46)$$

and it is a kind of 5-dimensional Schwarzschild black hole with a metric function as follows

$$f(r) = 1 - \frac{(N-3)Q^2}{(N-5)r^2}. \quad (47)$$



Unlike the case of  $N = 5$  the EYM black hole for  $N > 5$  is reminiscent of the Reissner-Nordstrom black hole in which the mass and charge terms are naturally separated. However there is a striking difference between EYM black hole in higher dimensions and the Reissner-Nordstrom, namely the gauge charge term in  $f(r)$  has the fixed power of  $r$  given by  $1/r^2$ . The surface gravity has the value

$$\kappa = \left| \frac{1}{2} f'(r_+) \right| = \left| \frac{1}{2} \frac{(N-3)m}{r_+^{N-2}} + \frac{(N-3)Q^2}{(N-5)r_+^3} \right| \quad (48)$$

$N > 5$

where  $r_+$  is the radius of the event horizon. The contribution by the parameters  $m, n, r_+$  and  $Q$  to the Hawking temperature (i.e.  $T_H = \frac{\kappa}{2\pi}$ , with all physical constants  $c, \hbar, G$ , and  $k$  set to unity) does not require any comment.

## 5 Conclusion

Our method employs the Wu-Yang Ansatz in higher dimensions which can be handled analytically and where the mass and gauge charge terms are taken separately. Unlike our approach, in the Ref.s [4-6], a position dependent mass density  $m(r)$  is assumed which is determined upon imposition of the EYM equations yielding the mass and constant charge terms. The EYM black hole has the marked distinction from the Reissner-Nordstrom black hole in higher dimensions as far as the  $r$  dependence of the charge term is concerned. Another significant difference of  $N=5$  and  $N>5$  black holes is that in the former case due to the logarithmic term we have both  $r_+$  and  $r_-$  (for possible black holes), whereas for  $N>5$  we have only  $r_+$ . The effect of the mass and charge on black hole formation, the existence of event (Cauchy for  $N=5$ ) horizons are plotted in Figs [2-5]. We note that the Wu-Yang Ansatz has been used also in finding black hole solutions in the EYM-Gauss-Bonnet theory [11].

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**Figure captions:**

Figure (1): Eq.s (19)-(20) (in the text) imply that for meaningful  $r_{\pm}$ ,  $m$  and  $Q$  must belong to a particular set.

Figure (2):  $r_{\pm}$  versus  $Q$  plot for different masses according to Eq.s (19)-(20). For large  $Q$  values, with fixed  $m$ ,  $r_{+}$  and  $r_{-}$  go to particular limits.

Figure (3): The plot of  $r_{\pm}$  versus  $m$  for different charges according to Eq.s (19)-(20). For large  $Q$ , irrespective of  $m$ ,  $r_{+}$  and  $r_{-}$  converge at constant values.

Figure (4):  $r_{+}$  versus  $Q$  plot for different masses (from Eq. (39)). For large  $Q$ , independent of  $m$ , it converges to a limit.

Figure (5):  $r_{+}$  versus  $m$  plot for different charges (from Eq. (39)). For large  $Q$ ,  $r_{+}$  becomes independent of  $m$ .

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